UW/PT–96–02
DOE/ER/40561-246-INT96-00-115
RU–96–9

Overlap formulation of Majorana–Weyl fermions.

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Abstract

An overlap method for regularizing Majorana–Weyl fermions interacting with gauge fields is presented. A mod(2) index is introduced in relation to the anomalous violation of a discrete global chiral symmetry. Most of the paper is restricted to 2 dimensions but generalizations to 2+8k dimensions should be straightforward.

In 2+8k dimensional Minkowski space the most basic fermion is the Majorana–Weyl (MW) fermion. MW fermions, when in irreducible real representations of Lie groups, can interact with the relevant gauge bosons. There exist gauge theories that cannot be viewed as containing only Weyl fermions. This is not true in four dimensions, and, for this reason, the overlap formalism was originally set up to deal only with the Weyl case [1]. In this paper we extend this formulation to MW fermions. For simplicity we work in 2 dimensions, but we believe that the main points generalize to other appropriate dimensions. A case of particular interest to us is ten dimensional N=1 supersymmetric Yang Mills; we hope to use a regulated version of this (non-renormalizable, anomalous) gauge theory as a means to regulate the four dimensional N=4 supersymmetric Yang Mills theory. The latter is interesting for many reasons, in particular, the large n limit of the $SU(n)$ case would have a cutoff independent critical value of the coupling constant where arbitrarily large planar Feynman diagrams dominate [2].

We work in Euclidean space and physical charge conjugation does not have a natural analogue [3]. What we mean then by MW is that a system of Weyl fermions in a real representation interacting with an external gauge field decouples into two isomorphic sub–systems and each such sub–system describes a set of MW fermions. Let the Weyl system be described by the action,

$$\mathcal{L} = \int \bar{\psi}_L(\sigma \cdot (\partial + iA))\psi_L \equiv \int \bar{\psi}_L C \psi_L,$$

with $A_\mu = -A_\mu^* = A_\mu^\dagger$ and $\sigma_\mu = (1, i)$ for $\mu = (1, 2)$. $\bar{\psi}_L$ and ψ_L are independent Grassmann variables. All group indices are suppressed. The purely imaginary character of A_μ (hermiticity is always assumed) is equivalent to a skew-symmetry of C: For any two ordinary complex functions ϕ_1 and ϕ_2, $\int \phi_1 C \phi_2 = -\int \phi_2 C \phi_1$. This skewness leads to the above mentioned decoupling and, consequently to a factorization of the determinant of C. Defining $\bar{\psi}_L = (\xi + i\eta)/\sqrt{2}$ and $\psi_L = (\xi - i\eta)/\sqrt{2}$ with ξ and η independent (real) Grassmann variables we find:

$$\mathcal{L} = \frac{1}{2}\int \xi C \xi + \frac{1}{2}\int \eta C \eta.$$

For fixed A_μ, correlation functions of the ψ's are given in a fixed pattern of sums of products of correlation functions of the ξ's and the η's. By analytic continuation to Minkowski space the standard expressions are obtained.

The Weyl theory has a $U(1)$ global (chiral) symmetry, potentially violated by quantum effects when the A_μ-fields are made dynamical. In the MW basis the $U(1)$ appears as an $O(2)$ rotating ξ and η into each other. If we go to a single MW system the $O(2)$ disappears

and all we are left with is a discrete Z_2 flipping the sign of ξ. In an anomaly free gauge theory one has to have both left–handed and right–handed fields and there will be several such Z_2's (some combinations of these Z_2's are not chiral). The chiral Z_2's forbid mass terms, and would imply the absence of bilinear fermionic condensates in finite Euclidean physical volumes. (The symmetries are discrete, so breaking them spontaneously in the infinite volume limit, even in two dimensions, is possible.)

However, the ancestry of these symmetries at the Weyl level indicates that, under certain circumstances, one should expect explicit violations of the global axial symmetries. For Weyl fermions the explicit breaking is caused by topologically nontrivial gauge backgrounds. The associated fermionic zero modes lead to non-vanishing symmetry-breaking condensates [4]. The stable part of the number of zero modes comes from certain C's that have a nonzero analytical index ($dim(Ker(C)) - dim(Ker(C^\dagger))$). Not for every type of gauge field does the possibility of a non-zero index even arise, but the option exists generically, and some gauge fields realize it. The above is well known.

In the MW case $C^\dagger = -C^*$ and the analytical index vanishes. One can still define a modulus 2 index in this case: The parity of the dimension of the kernel of C is invariant under small deformations of C subjected to skewness. We sketch a physicist's proof of this below.

We are on a compact manifold and A_μ is bounded; therefore $dim(Ker(C))$ is guaranteed to be a finite number, say k. We are interested in how k would change under a perturbation of C. Let the space C acts on be denoted by V. On $V \oplus V$ define $D = \begin{pmatrix} 0 & C \\ -C^* & 0 \end{pmatrix} = D^\dagger$. If $Ker(C) = span\ [u_1, u_2, ..., u_k]$, then, the kernel of D is given by:

$$span \left[\begin{pmatrix} 0 \\ u_1 \end{pmatrix}, \begin{pmatrix} 0 \\ u_2 \end{pmatrix}, ..., \begin{pmatrix} 0 \\ u_k \end{pmatrix}; \begin{pmatrix} u_1^* \\ 0 \end{pmatrix}, \begin{pmatrix} u_2^* \\ 0 \end{pmatrix}, ..., \begin{pmatrix} u_k^* \\ 0 \end{pmatrix} \right] \equiv span\ [\psi_\alpha,\ \alpha = 1, 2..., 2k].$$

Since D is hermitian we apply ordinary degenerate perturbation theory. We should diagonalize O, the perturbation matrix of D restricted to the kernel of D,

$$O_{\alpha,\beta} = \int \psi_\alpha^\dagger \delta D \psi_\beta.$$

The perturbation in D was induced by a perturbation in C, so O has the structure

$$O = \begin{pmatrix} 0 & A \\ -A^* & 0 \end{pmatrix},$$

with $A^t = -A;\ A_{ij} = \int u_i \delta C u_j$. Since the matrix A is antisymmetric its rank must be even. Hence, the perturbation will not be able to change the parity of k.

If the mod(2) index is odd ($(-)^k = -1$) there will be at least one zero mode stable under small deformations of the background gauge field and a bilinear condensate is possible, explicitly violating some of the chiral Z_2's. Conversely, an odd mod(2) index is a necessary condition for a non-vanishing bilinear expectation value in finite Euclidean volumes.

We now turn to the overlap formalism. Our objective is to find the analogous decoupling of a system of Weyl fermions in a real representation into MW fermions and understand how the discrete version of anomalous symmetry is realized. Any reasonable regularization of chiral gauge theories which preserves the bilinearity of the action like the overlap does, must provide equivalent realizations of decoupling and anomalous discrete symmetries.

We start from the Weyl overlap. The fermion induced action corresponding to a left–handed Weyl multiplet $z_W[A_\mu] = \int [d\bar{\psi}_L d\psi_L] e^{-\mathcal{L}}$ is replaced on the lattice by the overlap $z_W^{\rm lat}[U_\mu] = {}_U < L-|L+>_U$. The lattice is a symmetric torus of linear size l with l even. This ensures that the total number of MW degrees of freedom is even. The lattice link variables U_μ replace the continuum A_μ's and, in addition, implement the fermionic boundary conditions. The states $|L\pm>_U$ are the ground states of two many body hamiltonians $\mathcal{H}^\pm = \sum_{x\alpha,y\beta} a^\dagger_{x,\alpha} \mathbf{H}^\pm(x\alpha, y\beta; U) a_{y,\beta}$, $\{a^\dagger_{x,\alpha}, a_{y,\beta}\} = \delta_{\alpha,\beta}\delta_{x,y}$, $\alpha, \beta = 1, 2$ and $x = (x_1, x_2)$ with $x_\mu = 0, 1, 2, ..., l-1$. The single particle hermitian hamiltonians $\mathbf{H}^\pm$ are given by:

$$\mathbf{H}^\pm = \begin{pmatrix} \mathbf{B}^\pm & \mathbf{C} \\ \mathbf{C}^\dagger & -\mathbf{B}^\pm \end{pmatrix},$$

$$\mathbf{C}(x,y) = \frac{1}{2}\sum_\mu \sigma_\mu (\delta_{y,x+\mu} U_\mu(x) \ - \ \delta_{x,y+\mu} U^t_\mu(y)),$$

$$\mathbf{B}^\pm(x,y) = \frac{1}{2}\sum_\mu (2\delta_{x,y} \ - \ \delta_{y,x+\mu} U_\mu(x) \ - \ \delta_{x,y+\mu} U^t_\mu(y)) \ \pm \ m\delta_{x,y}.$$

The parameter m is restricted only by $0 < m < 2$. Group indices have been suppressed. The phases of the states $|L\pm>_U$ are chosen according to the Wigner-Brillouin convention [1]; they are irrelevant for the subsequent analysis.

First we wish to decouple the systems to attain a factorization of the overlap. Suppressing also the site indices we define a new set of creation/annihilation operators: $a_1 = \frac{\xi - i\eta}{\sqrt{2}}, a_2 = \frac{\xi^\dagger - i\eta^\dagger}{\sqrt{2}}$. The transformation is canonical. Under a Euclidean space rotation, both ξ and η transform as left–handed fields. Under a $U(1)$ phase transformation of the a_α's, ξ and η undergo an $O(2)$ transformation. The explicit form above ensures that the matrices $\mathbf{B}^\pm$ are real. A short computation shows that

$$\mathcal{H}^\pm = \frac{1}{2}\alpha^\dagger \mathbf{H}^\pm \alpha + \frac{1}{2}\beta^\dagger \mathbf{H}^\pm \beta,$$

with all indices suppressed, $\alpha_1 = \xi, \alpha_2 = \xi^\dagger$ and $\beta_1 = \eta, \beta_2 = \eta^\dagger$. The two terms above commute with each other, are isomorphic to each other and act in different spaces; thus the overlap will factorize into two equal factors.

We now turn to the mod(2) index. How would we solve for the overlap if we had a single MW multiplet and did not want to use the fact that it is the square root of a Weyl system ? To answer this question it is useful to change bases again: Define two hermitian operators, $\gamma_1 = \frac{\xi+\xi^\dagger}{\sqrt{2}}$ and $\gamma_2 = \frac{\xi-\xi^\dagger}{i\sqrt{2}}$. They obey the canonical commutation relations $\{\gamma_\alpha, \gamma_\beta\} = \delta_{\alpha,\beta}$. The collection of all the γ's, with all indices taken into account, generates a large Clifford algebra. Our hamiltonians are bilinears in this algebra given by

$$\mathcal{H}^{\pm}_{mw} = \frac{1}{2}\gamma \mathbf{H}^{\pm}{}_{mw}\gamma$$

with all indices suppressed. The hamiltonian matrices are purely imaginary and antisymmetric and take the form

$$\mathbf{H}^{\pm}{}_{mw} = \Gamma_3 \left(\Gamma_1 \mathrm{Re}\mathbf{C} + \Gamma_2 \mathrm{Im}\mathbf{C} + \mathbf{B}^{\pm}\right),$$

where we have introduced real symmetric two dimensional Dirac Γ-matrices, $\Gamma_1 = \sigma_3, \Gamma_2 = \sigma_1$ and picked $\Gamma_3 = -\sigma_2$, which is antisymmetric; the σ_i, $i = 1, 2, 3$ are the Pauli matrices. The $\mathbf{H}^{\pm}{}_{mw}$ are recognized as Γ_3 times lattice Wilson-Dirac operators, one with a positive mass term and the other with a negative one. This structure will generalize to other dimensions where MW fermions exist.

We need to carry out Bogolyubov transformations to bring the hamiltonians to some canonical form. This amounts to an orthogonal transformation which obviously preserves the anticommutation relations among the γ's. The $\mathbf{H}^{\pm}{}_{mw}$ can be brought to a canonical form where all nonzero elements are restricted to antisymmetric two by two blocks along the diagonal. One can arrange the left bottom entry of each block to be a positive number λ times i and order the blocks by the magnitude of λ (generically, there will be no degeneracies). One such two by two block would correspond to a term $i\lambda(\gamma_2\gamma_1 - \gamma_1\gamma_2)$. Here the γ's are the new, rotated, canonical generators. Rewriting the γ's in terms of new ξ's this term becomes $\lambda(\xi\xi^\dagger - \xi^\dagger\xi)$ indicating that in the ground state the appropriate ξ single particle state should be filled.

The Hilbert space splits into "even" and "odd" subspaces; on the even subspace the product of all the γ's in some fixed order (the "chirality") is ζ, while on the odd subspace this product is $-\zeta$ ($\zeta = \pm 1$ or $=\pm i$). The hamiltonians do not connect these two subspaces. There is then the possibility that one of the $|L\pm>_U$ ground states be odd and the other even. The way to ascertain whether this happens or not, is to look at the

combined orthogonal transformations which, individually, bring each hamiltonian matrix to its canonical form. The combined orthogonal matrix has a determinant equal to ± 1. If it is 1 the parities of the two ground states are the same; if it is -1 they are opposite. When they are opposite the overlap will vanish; in the continuum the vanishing would be attributed to C having an odd number of zero modes. If C has a single zero mode the expectation value of one fermion field could be non-zero. Similarly, in the lattice overlap formulation, the insertion of a single γ between the states will restore equal parity and will, generically, lead to a non-vanishing result.

Let us now explain why the sign of the determinant of the big orthogonal transformation, O, connecting the two bases in which the individual hamiltonians have their canonical forms indeed is the mod(2) index in the overlap formulation. Assume first that $\det(O) = 1$. Then, there exists an antisymmetric real matrix T such that $O = e^T$. The unitary operator $\mathcal{U}$ which implements the canonical transformation $\gamma' = O\gamma = \mathcal{U}^\dagger \gamma \mathcal{U}$ is given by $\mathcal{U} = e^{\frac{1}{2}\gamma T \gamma}$. If $|L- >_U$ is one of the ground states (why we pick the minus sign will become clear in the next paragraph, but is immaterial for the present discussion) the other is $|L+ >_U = \mathcal{U}^\dagger |L- >_U$. It is now made very explicit that the two states $|L\pm >_U$ have identical parities. Assume now that $\det(O) = -1$. Define a canonical transformation $\gamma' = \gamma_\sharp \gamma \gamma_\sharp = O_1 \gamma$ implemented by one particular $\gamma_\sharp$. The matrix O_1 is diagonal and has all entries -1 except the one associated with the chosen $\gamma_\sharp$ which is $+1$. It represents a "parity" transformation switching the odd and even "chiralities". Since the total space is even dimensional, $\det(O_1) = -1$. The unitary operator $\mathcal{U}$ implementing the canonical transformation defined by OO_1 is worked out just as before since $\det(OO_1) = 1$. The unitary operator implementing the original canonical transformation O is now $\mathcal{U}\gamma_\sharp$ and, just as above, we now arrive at the conclusion that the two ground states have opposite parities. In summary, the continuum $(-1)^{dim(Ker(C))}$ corresponds on the lattice to $\det(O)$. Note that on the lattice $(-1)^{dim(Ker(\mathbf{C}))} = 1$ always, showing that it would be mistaken to replace the continuum C by a finite matrix of rigid form.

For $\mathcal{H}^-_{mw}$ the ground state has the same parity for all gauge fields; the proof of this goes as follows. For very large values of $|m|$ the state is obviously independent of the gauge fields. Let us increase m towards some finite negative value. If the parity is to change at some mass value, some filled state must become empty at that value, at which point the associated λ vanishes. But we know already from the Weyl case that $\mathbf{H}^-$ always has a gap [1]. Therefore, as long as the mass is negative, the parity cannot change. We can choose the parity operator such that the ground state of $\mathcal{H}^-_{mw}$ is even.

It remains to be shown that there are actual instances in which $\mathcal{H}^+_{mw}$ has an odd ground state. This is trivial, since we could view the ordinary $U(1)$ Schwinger model [5]

as an $SO(2)$ gauge theory and each Weyl fermion as a doublet of Majorana ones of the same handedness. The instantons of $U(1)$, when viewed as $SO(2)$ configurations, make the required ground state odd. Of course, this is a very special case where MW fermions aren't necessary and there exists a more discerning index. However, embedding the $SO(2)$ into any $SO(n)$ in a trivial way shows that whenever the gauge group is $SO(n)$ and the fermions are in the vector (defining) representation of $SO(n)$ odd vacua will appear. Since this is the most general case the appearance of odd states is not an isolated event.

For odd states to appear in a way that has a chance to survive in the continuum limit it is necessary that the space of gauge fields in the continuum be disconnected. In two dimensions this will happen if the "true" gauge group, i.e. the one that is seen by the fermions, is multiply connected. Then this gauge group can be written as a simply connected (covering) group divided out by a nontrivial subgroup of its center. The center is always a subgroup of Z. Clearly, the "true" group for the fermions in the vector representation of $SO(n)$, $SO(n) = Spin(n)/Z(2)$, is of this type.

Another interesting example is provided by "adjoint QCD", where the "true" gauge group is $SU(n)/Z(n)$. (For $n = 2$, this is just the $SO(3)$ case above.) Let the Cartan generator given by $\frac{1}{\sqrt{n(n-1)}} diag(1, 1, 1, ..., 1, 1-n)$ be denoted by H, normalized by $tr(H^2) = 1$. Embedding a $U(1)$ instanton in $SU(n)/Z(n)$ in the H direction yields $n - 1$ zero modes for C [6]; these zero modes provide an $n - 1$ dimensional representation of the discrete symmetry of H which permutes the first $n - 1$ entries. We conclude that, for n even, odd ground states $|L+>_U$ will occur. A more thorough examination of this case is reserved for the future. The two immediate issues to be resolved are whether indeed there is a fundamental difference between odd and even n's [7] (this is important also for the large n limit), and whether for even n's larger than 2 one has non-vanishing bilinear condensates in finite Euclidean volumes. The overlap formulation is guaranteed to provide a numerical tool well suited to this problem and complementary to analytical methods. No other numerical approach we are aware of could be of similar use. We plan to address similar issues regarding gluino bilinear condensates in four dimensional N=1 pure supersymmetric Yang Mills theories.

Acknowledgments

This research was supported in part by the DOE under grants # DE-FG06-91ER40614 (PH and RN), # DE-FG06-90ER40561 (RN) and # DE-FG05-90ER40559 (HN).

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